**Tangent:** Let  and  be two neighboring points on a curve. As  tends to  (i.e. approaches to ) along the curve, the straight line joining  and  tends to definite straight line , called the tangent to the curve at .







Simply, we can say that a tangent is a straight line that touches a curve at a single point and does not cross through it. The point where the curve and the tangent meet is called the point of tangency.

**Theorem-01:** Derive the equation of the tangent to a curve  at any point of it.

**Answer:** Let  be the given point on the curve . Let  be a point neighboring to  on the curve.







Then the equation of the line  is



where  and are the current co-ordinates.

Now as , ,  then  and the line  tends to the tangent  to the curve at , so the equation of the tangent to the curve  at  is

.

**Theorem-02:** Derive the equation of the tangent to a curve  at any point of it.

**Answer:** The equation of the curve is 

The total differential of this curve is





We know the equation of the tangent at the point  is

From (1) and (2), we get



.

This is the equation of the tangent to the given curve at any point of it.

**Theorem-03:** Derive the equation of the tangent to a parametric curve  at any point of it.

**Answer:** The equation of the curve is  

Differentiating with respect to , we get

 and 

We know the equation of the tangent at the point  is

From (1), (2) and (3), we get

.

This is the equation of the tangent to the given curve at any point  of it.

**Tangents parallel to the co-ordinates axes:** If a tangent is parallel to the -axis , then  i.e.  and so we have  at that point.

If a tangent is parallel to the -axis, then  i.e.  and so we have  or  at that point.

**Angle of intersection of two curves:** If two curves intersect each other at , then the angle of intersection of curves is defined as the angle between the tangents to the curves at .













Let  and  be the angles which the tangents to the curve at  make with x- axis. Then if  be the required angle, it is evident from the figure,









.

This is the required angle.

If the two curves cut orthogonally, i.e. , then







.

If the equation of the curves be  and , then

.

This curves cut orthogonally i.e.  if .

**Problem-01:** Find the equation of the tangent at  to the curve 

**Solution:** The equation of the curve is .

Here  

Differentiating (1) partially with respect to  and, we get



and 

The equation of the tangent is,











.

**Problem-02:** Find the equation of the tangent to the curve  at .

**Solution:** The equation of the curve is . 

Differentiating (1) with respect to , we get



At , the slope of the given curve is



The equation of the tangent at is,









Replacing  by , we get

.

**Problem-03:** Find the equation of the tangent to the curve  at the point .

**Solution:** The equation of the curve is .

Here  

Differentiating (1) with respect to , we get



and 

The equation of the tangent at  is,













.

**Problem-04:** Find the angle of intersection of the curves .

**Solution:** The equations of the curves are,

and  

Adding (1) and (2), we get





Subtracting (1) from (2), we get





Therefore the point of intersection of the curves is .

The equations (1) and (2) can be written as,

and  

Differentiating (3) and (4) with respect to , we get



If be the angle of intersection of the curves, then













.

**Problem-05:** Find the condition that the conics will cut orthogonally.

**Solution:** The equations of the conics are,

and  

Differentiating (1) and (2) with respect to , we get



The curves will cut orthogonally at  if





Since the point  is common to both (1) and (2), the required condition is obtained by eliminating  from (1), (2) and (3).

Subtracting (2) from (1), we get

Comparing (3) and (4), we get





which is the required condition.

**Problem-06:** Find the condition that the straight line  may touch the curve .

**OR**

If touches the curve , then show that .

**Solution:** The equations of the line and curve are,

and  

The curve (2) can be written as,

Differentiating (3) partially with respect to , we get



and 

The equation of the tangent at is,









. 

If (1) touches (2), then (1) and (4) must be identical. So from (1) and (4) we can write,











.

This is the required condition. **(Showed)**

**Problem-07:** Show that, the condition that the curves  and may touch if .

**Solution:** The equations of the curves are,

and  

Differentiating (1) partially with respect to , we get

 and 

The equation of the tangent at is,









. 

Again differentiating (2) partially with respect to , we get

 and 

The equation of the tangent at is,









. 

Since the given curves (1) and (2) touch to each other at the point  so that the tangents (3) and (4) represent the same straight lines. Hence from (3) and (4), we get







Adding these terms, we have





. **(Showed)**

**Homework:**

**Problem-01:** Find the equation of the tangent at  to the curve 

**Problem-02:** Find the equation of the tangent to the curve  at the point .

**Problem-03:** Prove the condition that the straight line  touches the curve  is

.

**Normal:** The normal to a curve at a point is the line perpendicular to the tangent line at the point.











**Theorem-04:** To find the equation of the normal to a curve  at any point of it.

**Answer:** We know that the tangent to a curve  at any point  is

where  is the gradient of tangent.

Also any line through the point is given by

If this line is normal to the curve  at  then this must be perpendicular to the tangent to the curve, so that



Now from (2) and (3), we get

.

This is the equation of the normal to the given curve at any point of it.

**Theorem-05:** To find the equation of the normal to a curve  at any point of it.

**Answer:** The equation of the curve is 

The total differential of this curve is





We know the equation of the normal at the point  is

From (1) and (2), we get



.

This is the equation of the normal to the given curve at any point of it.

**Theorem-06:** Derive the equation of the tangent to a parametric curve  at any point of it.

**Answer:** The equation of the curve is  

Differentiating with respect to , we get

 and 

We know the equation of the normal at the point  is

From (1), (2) and (3), we get

.

This is the equation of the normal to the given curve at any point  of it.

**Problem-01:** Find the equation of the normal at  to the curve 

**Solution:** The equation of the curve is 

Here  

Differentiating (1) partially with respect to  and, we get

 and 

The equation of the normal at  is,









At  the equation of the normal is,





Replacing X and Y by x and y, we get

.

This is the required equation.

**Problem-02:** Show that the normal at any point of the curve  is at a constant distance from the origin.

**Solution:** The equation of the curve is .

Here  

Differentiating (1) with respect to , we get



and 

The equation of the normal at  is,















Replacing X and Y by x and y, we get



Now the distance of this normal from the origin is



Thus the distance of the normal from the origin is constant. **(Showed)**

**Homework:**

**Problem-01:** Find the equation of the normal at  to the curve Also find normal at the point where it cuts y-axis.

**Problem-02:** Find the tangent and the normal to the curve  at the point .

**Problem-03:** Find the tangent and the normal to the curve  at the point .

**Problem-04:** Find the tangent and the normal to the curve  at the point .

**Question:** Define Cartesian subtangent, subnormal, lengths of tangent and normal.

**Answer:** Let  be any point on the curve. Let the tangent and normal at  to the curve meet  -axis in  and respectively. Let  be the ordinate, then . Then is called the subtangent, is called the subnormal and the lengths of tangent intercepted between the point of contact and  -axis, i.e. is called the length of tangent. Similarly, the length of the normal intercepted between the point of contact and  -axis, i.e.is called the length of normal.





















If the angle which the tangent at makes with  -axis be , then  and from the figure we get the following results:

1. Subtangent = 
2. Subnormal = 
3. Length of the tangent= 
4. Length of the normal= .

**Polar co-ordinates:** Let  be a fixed point and  be a fixed straight line through . This fixed point  is called pole and the fixed straight line  is called initial line. Then the position of a point  on the plane is given by:

1. its distance  from the pole and
2. the inclination  of  to the initial line .















Here is called the polar co-ordinate of the point , is called the radius vector and  is called the vectorial angle of the point .

**Relation between Polar and Cartesian co-ordinates:** Let the Cartesian co-ordinates of the point  be . Also the polar co-ordinates of the same point be .















From the figure we have





and 

.

**Angle between radius vector and tangent:** Let  and  be two neighboring points on the curve . Let  is the pole and  is the initial line. Join ,  and . From  draw  perpendicular to .  is the tangent to the curve at the point .

Let  be the angle between the tangent  and the radius vector .

Now as ,, , the secant the tangent  and .

































 , Neglecting higher powers of infinitesimals.

.

**Question:** If be the length of perpendicular from pole, then show that .

**Answer:** Let  be any point on the curve . Let  is the pole and  is the initial line. Join . From  draw  perpendicular to the tangent at  to the curve. Let  and  .

















From , we get







which gives the length of the perpendicular from pole on the tangent.

If we require the value of  in terms of  and , then we can proceed as follows:









. **(Showed)**

**Polar equation:** An equation for a curve written in terms of the polar coordinates  and .

**Pedal equation:** The relation between  and  for a given curve is called its pedal equation, where  is the length of the perpendicular from the pole on the tangent to the curve at any point of it and  is the radius vector of this point.

**Question:** To find the pedal equation from Cartesian equation and polar equation.

**Answer:** **1st part (Cartesian equation):** The equation of the tangent at any point  is





Since is the length of the perpendicular from the pole on the tangent to the curve, so

Also since  is the radius vector of the point , so

And let the Cartesian equation of the curve be

Now eliminating  and  from (1), (2) and (3), we get a relation  and , which is the required pedal equation of the curve.

**2nd part (Polar equation):** Let the curve be  

Also we know that

Now eliminating  from (1) and (2), we get a relation  and , which is the required pedal equation of the curve.

Alternatively, Let the curve be  

Also we know that

and  

Now eliminating  and  from (1), (2) and (3), we get a relation  and , which is the required pedal equation of the curve.

**Problem-01:** Find the pedal equation of the parabola .

**Solution:** The equation of the parabola is . 

Differentiating (1) with respect to , we get



The equation of the tangent at  is,







Since is the length of perpendicular from  on the tangent (2), so









Also we have 

.

This is the required pedal equation.

**Problem-02:** Find the pedal equation of the astroid 

**Solution:** The equation of the parabola is 

Here  

Differentiating (1) partially with respect to  and, we get

 and 

The equation of the tangent at  is,











. 

Since is the length of perpendicular from  on the tangent (2), so





Also we have 







.

This is the required pedal equation.

**Problem-03:** Show that the pedal equation of the ellipse  is .

**Solution:** The equation of the ellipse is . 

Here  

Differentiating (2) partially with respect to  and, we get

 and 

The equation of the tangent at  is,









. 

Since is the length of perpendicular from  on the tangent (3), so





Also we have



Now the pedal equation of the given ellipse will be obtained by eliminating  and  from (1) , (4) and (5). Eliminating  and  from (1) , (4) and (5), we get



. **(Showed)**

**Problem-04:** Show that the pedal equation of the cardioid  with respect to pole is 

**Solution:** The given curve is . 

Differentiating (1) with respect to , we get



We know,













.

Also we know that











. **(Showed)**

**Problem-05:** Find the pedal equation of .

**Solution:** The given curve is . 

Taking logarithm on both side, we have

Differentiating (2) with respect to , we get







We know,









Also we know that









.

This is the required pedal equation.

**Homework:**

**Problem-01:** Find the pedal equation of the circle 

**Problem-02:** Find the pedal equation of the curve 

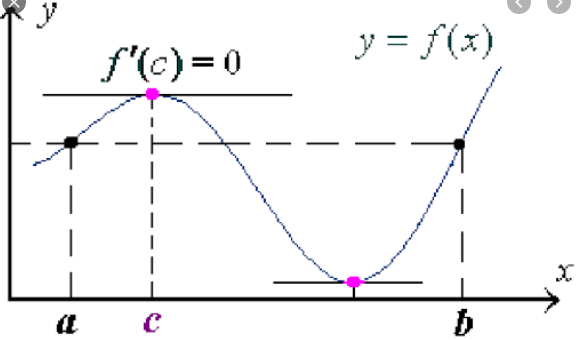
**Problem-03:** Find the pedal equation of .

**Theorem-07:** State and prove Rolle’s Theorem.

**Statement:** If a function  is such that

1. it is continuous in the closed interval 
2. it is differentiable in the open interval 
3. 

then there exists at least one value in the interval  such that .



**Proof:** Since the function  is continuous on the closed interval  and , so when  then the function should either increase or decrease unless  is constant.

If the function  is constant function on then we have

.

In particular when .

Hence the theorem is true for any .

Again suppose the function  increases when  and , so the function  increase at the point  where  such that  and the function  decreases thereafter.

At the point  where , there is a maximum value of the function , which is . So by the definition of maximum value, we have

 and , where and 

 and 

 and 

 and .

Since  derivable in the open interval , so by the definition of differentiability we have



Here this statement will be valid if . Hence the theorem is true for any . **(Proved)**

**Algebraic interpretation of Rolle’s Theorem:** If  be a polynomial in  and ,  be the two roots of the equation , then from Rolle’s Theorem we find that at least one root of the equation  lies between  and .

**Theorem-08:** State and prove Mean value Theorem.

**Statement:** If a function  is such that

1. it is continuous in the closed interval 
2. it is differentiable in the open interval ,

then there exists at least one value in the interval  such that .

**Proof:** Consider the function  

where A is a constant to be determined such that







Now  is given to be continuous in the closed interval  and is differentiable in the open interval .

Also, A being a constant,  is also continuous in the closed interval  and is differentiable in the open interval .

So  is

1. continuous in the closed interval 
2. differentiable in the open interval  and also
3. .

Therefore,  satisfies all the three conditions of Rolle’s Theorem. So there must exist at least one value in the interval  such that





Now from (2) and (3) we get

. **(Proved)**

**Problem-01:** Verify Rolle’s Theorem for the function  in .

**Solution:** The given function is 

i.e.  

Differentiating (1) with respect to , we get



Since  and  are continuous for all  and  for any finite value of , so  is continuous in  and  is derivable for all values of in .

Also  and  so that .

Since  satisfies all the three conditions of Rolle’s Theorem, so there exists at least one number ‘c’ in such that









.

Hence the Rolle’s Theorem is verified.

**Problem-02:** Verify the truth of Rolle’s Theorem for the function  in .

**Solution:** The given function is  

Differentiating (1) with respect to , we get



Since is a polynomial in  , so  is continuous and differentiable in the interval .

Therefore,  is continuous in the interval  and differentiable in the interval .

Here 

and 

.

Since  satisfies all the three conditions of Rolle’s Theorem, so there exists at least one number ‘c’ in such that





.

Hence the Rolle’s theorem is verified.

**Problem-03:** Verify the truth of Rolle’s Theorem for the function .

**Solution:** The given function is  

Differentiating (1) with respect to , we get

.

Since is a polynomial in  , so  is continuous and differentiable in the interval .

Now  gives,





.

i.e.  and .

Thus we find that .

Also,  is continuous in the interval  and differentiable in the interval .

Since  satisfies all the three conditions of Rolle’s Theorem, so there exists at least one number ‘c’ in such that





.

Hence the Rolle’s theorem is verified.

**Problem-04:** Justify the validity of the Mean Value Theorem for the function  in the interval .

**Solution:** The given function is  

Differentiating (1) with respect to , we get



Since is a polynomial in  , so  is continuous and differentiable everywhere.

Therefore,  is continuous in the interval  and differentiable in the interval . Thus the hypotheses of the Mean Value Theorem with  and .

Here 

and 

Now by Mean Value Theorem there exists at least one number ‘c’ in the interval  such that









.

Hence the Mean Value Theorem is valid for the given function.

**Problem-05:** Justify the validity of the Mean Value Theorem for the function  in the interval .

**Solution:** The given function is  

Differentiating (1) with respect to , we get



Since is a polynomial in  , so  is continuous and differentiable everywhere.

Therefore,  is continuous in the interval  and differentiable in the interval . Thus the hypotheses of the Mean Value Theorem with  and .

Here 

and 

Now by Mean Value Theorem there exists at least one number ‘c’ in the interval  such that











.

Hence the Mean Value Theorem is valid for the given function.

**Homework:**

**Problem-01:** Verify Rolle’s Theorem for the function  in .

**Problem-02:** Verify Rolle’s Theorem for the function  in .

**Problem-03:** Verify Rolle’s Theorem for the function  in .

**Problem-04:** Justify the validity of the Mean Value Theorem for the function  in the interval .

**Increasing function:** A function defined on an interval where , is called an increasing function over the interval if .

**Example:** is an increasing function.

**Decreasing function:** A function defined on an interval where , is called a decreasing function over the interval if .

**Example:**  is a decreasing function.

**Concavity and Convexity (with respect to a given point):**











**Fig. (i)** **Fig. (ii)**

Let  be the tangent to a curve at . Then the curve at  is said to be concave or convex with respect to a point  not lying on , according as a small portion of the curve in the immediate neighbourhood of (on both side of it) lies entirely on the same side of  as [as in Fig. (i)], or on opposite sides of  with respect to [as in Fig. (ii)].













**Fig. (iii)**

Thus, in Fig.(iii) the curve at  is convex with respect to , and concave with respect to or . The curve at  is concave with respect to . Again, the curve at  is convex to  and concave to .

**Point of Inflection:**







**Fig. (i)**

In some curves, at a particular point on it, the tangent line crosses the curve, as in Fig. (i). At this point, clearly the curve, on one side of , is convex, and on the other side it is concave with respect to any point (not lying on the tangent line). Such a point on a curve is defined to be a point of inflection (or a point of contrary flexure).

**Analytical Test of Concavity or Convexity (with respect to the x-axis):**

































**Fig. (i) Fig. (ii)**

Let  be a point on the curve ,  a neighbouring point whose abscissa is  (being small, positive or negative). Let be the tangent at , and let the ordinate  of  intersect  at .

The equation to  is



and abscissa  of  being ,its ordinate

.

Also the ordinate of  is



. 

Now assuming  to be continuous at  and  there,  has the same sign as that of  when  is sufficiently small.

Hence from (1),  has the same sign as that of , for positive as well as negative values of , provided it is sufficiently small in magnitude.

**Firstly,** let the ordinate  or  be positive.

If  is positive, from (1)  for  on either side of  in its neighborhood, and so the curve in the neighborhood of  (on either side of it) is entirely above the tangent i.e. on the side opposite to the foot on the x-axis of the ordinate , as in Fig.(i). Hence, the curve at  is convex with respect to the x-axis.

Again if  is negative, from (1)  for  on either side of , and so the curve in the near  is entirely below the tangent i.e. on the same side of , as in Fig.(ii). Hence, the curve at  is concave with respect to the x-axis.

**Secondly,** let the ordinate  or  be negative.





































**Fig. (i) Fig. (ii)**

If  is positive, from (1)  for  on either side of , and as both are negative, is numerically less than , as in Fig.(i). The curve, therefore, at  lies on the same side as  is with respect to the tangent . Hence, the curve at  is concave with respect to the x-axis.

Again if  is negative, we similarly get the curve at  convex with respect to the x-axis, as in Fig. (ii).

Combining the two cases, we get the following criterion for convexity or concavity of a curve at a point with respect to the x- axis:

If  is positive at , the curve at  is convex to the x-axis.

If  is negative at , the curve at  is concave to the x-axis.

**Analytical condition for point of inflection:** Let  be a point on the curve ,  a neighboring point whose abscissa is  (being small, positive or negative). Let be the tangent at , and let the ordinate  of  intersect  at .



















**Fig.(i)**

The equation to  is



and abscissa  of  being ,its ordinate

.

Let  at , and .

Then the ordinate of  is



,

and the sign of this for sufficiently small  is the same as that of , which has got opposite signs for positive and negative values of , whatever be the sign of at . Thus, near  the curve is above the tangent on one side of , and below the tangent on the other side, as in Fig. (i). Hence,  is a point of inflection.

Thus, the condition that  is a point of inflection on the curve is that, at ,

 and .

**Problem-01:** Examine the curve  regarding its convexity or concavity to the x- axis, and determine its point of inflection, if any.

**Solution:** The given curve is,  

Differentiating (1) with respect to , we get







Here equation (2) is negative for all values of  excepting those which make , i.e. ,  being any integer , positive or negative.

Thus, the curve is concave to the x-axis at every point, excepting at points where it crosses the x-axis.

At these points, given by , we have

 and .

Hence, those points where the curve crosses the x-axis are points of inflection.

**Problem-02:** Show that the curve  is concave to the foot of the ordinate everywhere except at the origin.

**Solution:** The given curve is,  

The curve can be written as,  

Differentiating (2) with respect to , we get





Here equation (3) is negative for all values of  excepting at the origin. Thus, the curve is concave to the foot of the ordinate everywhere except at the origin.

**Problem-03:** Prove that  is a point of inflection of the curve .

**Solution:** The given curve is,  

Differentiating (1) with respect to , we get





At , we have

 and .

Hence the point  is a point of inflection. **(Showed)**

**Homework:**

**Problem-01:** Prove that the curve  is convex to the x-axis at every point.

**Problem-02:** Prove that the curve  is everywhere concave to the y-axis excepting where it

crosses the y-axis.

**Problem-03:** Show that the curve  is convex to the foot of the ordinate in the range ,

and concave where . Show also that the curve is convex everywhere to the y-axis.

**Problem-04:** Prove that the origin is a point of inflection of the curves  and .

**Maximum (relative maximum or local maximum) value of a function:** A function is said to have a maximum value atif for all values of in the open interval , where  is a small positive number.

**Minimum (relative minimum or local minimum) value of a function:** A function is said to have a minimum value atif for all values of in the open interval , where  is a small positive number.

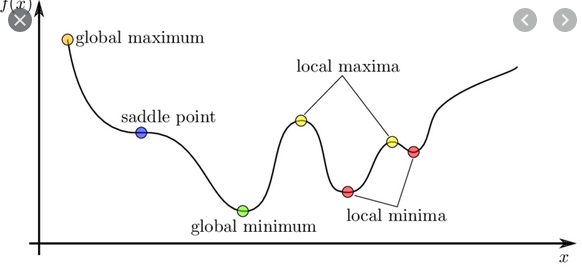
**Global (absolute) maximum value of a function:** A function  is said to have a global maximum value at if for all values of in the domain of the function.

**Global (absolute) minimum value of a function:** A function  is said to have a global minimum value at if for all values of in the domain of the function.

**Critical point:** A point on a curve in which derivative is zero or function is not differentiable.

**Stationary point:** A stationary point is a point on the curve where gradient of a function is zero. If gradient of the curve changes sign at stationary point then it called turning point otherwise horizontal Inflection.

**Saddle point:** A saddle point is a point in the domain of a function which is a stationary point but not a local extremum.

****

**Theorem-07:** State and prove Fermat's Theorem for finding maxima and minima.

**OR**

State and prove interior extremum theorem.

**Statement:** If has maximum or minimum value at  and  exists, then .

**Proof:** Let us consider that  is a function of  with the assumption that  is continuous and derivable and finite for all values of  in the neighborhood of .

At  the value of  is .

Consider two values of , namely  and  in the neighborhood and on either side of ,  being very small.

By Taylor’s theorem we get,



where  is the remainder after  terms.

Here  is very small so that by neglecting 2nd and higher degree term we get,



Again replacing  by  in (1), we get

We know that for maximum or minimum the sign of and  must be the same. So from (1) and (2), we conclude that if and  have the same, then  otherwise they have different signs.

Hence, if  has maximum or minimum value at  and  exists, then . **(Proved)**

**Theorem-08:** State and prove the sufficient conditions for the existence of extreme values of a function.

**Statement:** If is continuous at ,  and  then

1.  has maximum value  at , if 
2.  has minimum value  at , if .

Again if  then

1. for maximum and minimum,  must be an even number and  should be negative for maximum and positive for minimum.

**Proof:** Let us consider that  is a function of  with the assumption that  is continuous and derivable and finite for all values of  in the neighborhood of .

At  the value of  is .

Consider two values of , namely  and  in the neighborhood and on either side of ,  being very small.

By Taylor’s theorem we get,

where  is the remainder after  terms.

Here  is very small so that by neglecting 3rd and higher degree terms we get,





Again replacing  by  in (2), we get

We know that for maximum or minimum the sign of and  must be the same. From (2) and (3), since , so we conclude thatand  have the same.

When  is maximum value then from (1) and (2), by the definition we have



and 

Hence  has maximum value  at , if .

Again when  is minimum value then from (1) and (2), by the definition we have



and 

Hence  has minimum value  at , if .

Again if  then neglecting 4th and higher degree terms we get from (1),

Again replacing  by  in (4), we get

For maximum or minimum the sign of and  must be the same. So from (4) and (5) if we conclude thatand  have the same sign then  otherwise they have different signs.

Again neglecting 5th and higher degree terms we get from (1),

Again replacing  by  in (6), we get

For maximum or minimum the sign of and  must be the same. So from (6) and (7), since , so we conclude thatand  have the same sign.

When  is maximum value then from (6) and (7), by the definition we have



and 

Hence  has maximum value  at , if .

Again when  is minimum value then from (6) and (7), by the definition we have



and 

Hence  has minimum value  at , if .

Proceeding in this way we find that in general:

If  then for maximum and minimum,  must be an even number and  should be negative for maximum and positive for minimum. **(Proved)**

**Working rule for finding maxima and minima:**

If function be given, find and equate it to zero. Solve this equation for.

Let its roots be, … … ….

Find and hence find …. …. …. …..

* If is negative we have a maximum at .
* If is positive we have a minimum at .
* If find and then .
* If , then there is neither maxima nor minima at 

If ; find and then .

If is negative, then is maximum and if is positive, then is minimum at.

If Then find and so on.

* 
* if be odd, then there is neither maxima nor minima at 
* if be even and if is negative then is maximum at and
* ifis positive then is minimum at .

**Problem-01:** Find the maximum and minimum values of .

**Solution:** The given function is,

Differentiating with respect to *x* we get,

We know that for maximum and minimum values,













Again, differentiating eq.(2) with respect to *x* we get,



For  we get,



Therefore, the given function is maximum at 

The maximum value is,



For  we get,



Therefore, the given function is minimum at 

The minimum value is,



For  we get,



Therefore the test fails.





Therefore, the given function is neither maximum nor minimum at ****

**Problem-02:** Find the extremmum values of .

**Solution:** The given function is,

**** 

Differentiating with respect to *x* we get,

****

We know that for maximum and minimum values,













Again, differentiating eq.(2) with respect to *x* we get,

****

For  we get,



Therefore, the given function is maximum at 

The maximum value is,



For  we get,

****

Therefore, the given function is minimum at 

The minimum value is,

 (*Ans*).

**Problem-03:** Investigate for what values of  the function  is a maximum or minimum. Find also the maximum and minimum values.

**Solution:** The given function is,

**** 

Differentiating with respect to *x* we get,

**** 

We know that for maximum and minimum values,













Again, differentiating eq.(2) with respect to *x* we get,

****

For  we get,



Therefore, the given function is minimum at 

The minimum value is,



For  we get,

****

Therefore, the given function is maximum at 

The maximum value is,



**Problem-04:** Show that  is a maximum at . Deduce that .

**Solution:** The given function is,

**** 

Differentiating with respect to *x* we get,

**** 

We know that for maximum and minimum values,













Again, differentiating eq.(2) with respect to *x* we get,

****

For  we get,







Since  is negative at , so the given function is maximum at . **(Showed)**

The maximum value is,

.

2nd part: Since  is maximum for  so

****

****

**** **(Deduced)**

**Problem-05:** Find the maximum and minimum values for .

**Solution:** The given function is,

Differentiating with respect to *x* we get,

**** 

We know that for maximum and minimum values,











Again, differentiating eq.(2) with respect to *x* we get,

****

For  we get,



Since  is positive at , so the given function is minimum at .

The maximum value is,

.

For  we get,









.

Since  is negative at , so the given function is minimum at .

The maximum value is,





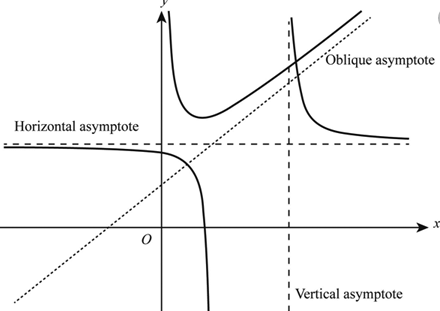




**Homework:**

1. For what values of  the expression  is maximum or minimum? Find also the maximum and minimum values.
2. Examine  for maximum or minimum values.
3. Find maximum or minimum value of the function.
4. Find maximum or minimum value of the function .
5. Show that  has neither a maximum nor a minimum.
6. Show that the maximum value of  is .
7. Show that the maximum value of  is less than its minimum value.
8. Show that the maximum value of  is .
9. Show that the minimum value of  is .
10. Show that the maximum value of  is .
11. Examine whether  possesses a maximum or a minimum and determine the same.
12. Investigate for what values of  the function  is a maximum or minimum. Find also the maximum and minimum values.

**Asymptotes:** An asymptote of a [curve](https://en.wikipedia.org/wiki/Curve)  is a straight line such that the distance between the curve and the line approaches zero as one or both of the *x* or *y* coordinates [tends to infinity](https://en.wikipedia.org/wiki/Limit_of_a_function#Limits_at_infinity).



There are three types of asymptotes:

1. Vertical asymptotes,
2. Horizontal asymptotes and
3. Oblique asymptotes.

**Question:** If asymptotes not parallel to y-axis are given by , where  is any of the real finite roots of , then show that for each value of ,

.

**Answer:** the general form of the equation of an algebraic curve of the  degree (arranging in groups of homogeneous terms) is,



The equation (1) also can be written as,

where  is an algebraic polynomial of degree .

Dividing equation (2) by , we get

Now if  ne an asymptote, where and  are finite, then



Hence from (3), making , since is finite, and the functions ,, ,etc. which are algebraic polynomials in  are accordingly finite, we get

.

Again, since in this case,

.

We can write



where  is a function of  such that  when .

Thus, 

From (3), we get

Expanding each term by Taylor’s theorem and using , we get







Now multiplying throughout by and making , we get



This is the required result. **(Showed)**

**Working rule for asymptotes of algebraic curves:** For an algebraic curve of the nth degree, first of all see if the term involving  is absent, in which case, the coefficient of the highest power of involved in the equation (unless it is merely a constant independent of ) equated to zero will give asymptotes parallel to the y-axis.

Similarly, if the term involving  is absent, the coefficient of the highest available power of  equated to zero will in general give asymptotes parallel to the x-axis.

Next, replacing  by 1 and  by in the homogeneous nth degree terms, get . Similarly, get from the  degree terms, and if necessary from the  degree terms and so on. Now equating  to zero, obtain the real finite roots , , etc. which will indicate the directions of the corresponding asymptotes (repeated roots giving in general a set of parallel asymptotes).

For each non-repeated root (, say), a definite value  of



is obtained, and the corresponding asymptote  determined.

**Special cases:**

(1). If for any  we get , then there is no asymptote corresponding to this value.

(2). If for any  we get , then 

will give two values (say ,) of in general, and thereby giving two parallel asymptotes of

the type  and .

**Asymptotes of Polar curves:** Let  be the polar equation to a curve. This may be written as,

. According to this assumption the formula for asymptotes is , where  is the solution of .

**Problem-01:** Find the asymptotes of the cubic .

**Solution:** The given curve is,

This is an algebraic curve of third degree. Since the terms  and  are both present, so there are no asymptotes parallel to either the x-axis or the y-axis. It has oblique asymptotes of the type,

Putting 1 for x and m for y in all 3rd order terms of (1), we get





Again putting 1 for x and m for y in all 2nd order terms of (1), we get



Now  gives,











.

Therefore, the values are, .

We know 

For , form (3) we get, .

For , form (3) we get, .

For , form (3) we get, .

Using and in (2) we have

.

Using and in (2) we have

.

Using and in (2) we have

.

The required asymptotes are, , and .

**Problem-02:** Find the asymptotes of .

**Solution:** The given curve is,



This is an algebraic curve of third degree. Since the terms  and  are both absent, so there are asymptotes parallel to the x-axis and the y-axis.

The coefficient of the term  is .

The asymptote parallel to the y-axis is

.

The coefficient of the term  is .

The asymptote parallel to the x-axis is

.

It has oblique asymptotes of the type,

Putting 1 for x and m for y in all 3rd order terms and in all 2nd order terms of (1), we get





Again putting 1 for x and m for y in all 2nd order terms of (1), we get



Now  gives,





.

Therefore, the values are, .

We know 

For , form (3) we get, .

For , form (3) we get, .

Using and in (2) we have

.

Using and in (2) we have

.

The required asymptotes are, , and .

**Problem-03:** Find the asymptotes of the Folium of Descartes .

**Solution:** The given curve is,



This is an algebraic curve of third degree. Since the terms  and  are both present, so there are no asymptotes parallel to either the x-axis or the y-axis. It has oblique asymptotes of the type,

Putting 1 for x and m for y in all 3rd order terms of (1), we get





Again putting 1 for x and m for y in all 2nd order terms of (1), we get



Now  gives,









Therefore, the only real value of m is, .

We know 

For , form (3) we get, .

Using and in (2) we have

.

The required asymptote is, .

**Problem-04:** Find the asymptotes of .

**Solution:** The given curve is,

This is an algebraic curve of third degree. Since the term  is present so there is no asymptote parallel to the x-axis and since is absent, so there is an asymptote parallel to the y-axis.

The coefficient of the term  is .

The asymptote parallel to the y-axis is

.

It has oblique asymptotes of the type,

Putting 1 for x and m for y in all 3rd order terms of (1), we get







Since 2nd order term is absent so





Again putting 1 for x and m for y in all 1st order terms of (1), we get



Now  gives,

.

Therefore, the values are, .

Since the roots are repeated so in the case of repeated roots we have

For , substituting the values of , and in (3), we get,









Using and in (2) we have

.

Using and in (2) we have

.

The required asymptotes are, , and .

**Problem-05:** Find the asymptotes, if any, of the curve .

**Solution:** The given curve is,



Let 



The directions in which  are given by









Now differentiating (2) with respect to , we get



Using  in (3), we get



The required asymptote is given by,





.

**Homework:**

**Problem-01:** Find the asymptotes of .

**Problem-02:** Find the asymptotes of .

**Problem-03:** Find the asymptotes of .

**Problem-04:** Find the asymptotes of .

**Problem-05:** Find the asymptotes of .

**Problem-06:** Find the asymptotes, if any, of the curve .

**Problem-07:** Find the asymptotes, if any, of the curve .

**Problem-08:** Find the asymptotes, if any, of the curve .

**Answers:** **P-1:** ; **P-2:** .

**P-3:** ; **P-4:** .

**P-5:** ; **P-6:** ; **P-7:** ; **P-8:** .

**Curvature:** Let  be a given point on a curve, and be a point on the curve near . Let the arc  measured from some fixed point  on the curve be  , and the arc  be ; then arc . Let , be the tangents to the curve at  and , and let  and ; then . Thus,  is the change in the inclination of the tangent line as the point of contact of the tangent line describes the arc .























The quotient  is called the average curvature of the arc .

The curvature at  is the limiting value, when it exists, of the average curvature when  (from either side) along the curve,

i.e. 

.

Thus, the curvature is the rate of change of direction of the curve with respect to the arc, or roughly speaking, the curvature is the “rate at which the curve curves”.

The reciprocal of the curvature at any point  is called the radius of curvature at . It is denoted by  and defined as,

.

**Envelopes:** If each of the members of the family of curves  touches a fixed curve , then  is called the envelope of the family of curves .The curve  also, at each point, is touched by some member of the family .